Manin-Mumford, André-Oort, the equidistribution point of view.

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1 Introduction

These notes were prepared for the 2005 Summer School "Equidistribution in Number theory" organized by Andrew Granville and Zeev Rudnick in Montreal. It's a pleasure to thank them for the opportunity of giving these lectures. The aim of this text is to describe the conjectures of Manin-Mumford, Bogomolov and André-Oort from the point of view of equidistribution. This includes a discussion of equidistribution of points with small heights of CM points and of Hecke points. We tried also to explain some questions of equidistribution of positive dimensional "special" subvarieties of a given variety.

The assignment by the organizer was to try to present a large overview but to avoid using technical language. For example I was not allowed to use the following notions (I quote the organizers):

1. Adeles (avoid them like the plague!)
2. Shimura varieties
3. Semisimple groups
4. Arakelov theory

I was unable to fill out all the requirements but I tried to focus on significant examples and the presentation of the general picture in a coherent view. Complete proofs are given in only a small amount of simplified cases when it can help the reader improve his intuition on the general case. The main statements are only sketched and the material of these notes covers much more than it's possible to present in a few hours of lectures. We hope that this text will complement the lectures and will be of some help for the reader interested in the understanding of these topics in a deeper way.

2 Informal examples of equidistribution.

2.1 Preliminary results from measure theory

Let X be a metric space and $\mathcal{P}(X)$ the set of Borel probability measures on X. Let $C(X)$ be the set of bounded continuous functions on X. We say that a sequence $\mu_n \in \mathcal{P}(X)$ is weakly convergent to $\mu \in \mathcal{P}(X)$ if for all $f \in C(X)$

$$\mu_n(f) = \int_X f \, d\mu_n \longrightarrow \mu(f) = \int_X f \, d\mu \quad \text{as } n \to \infty.$$

We'll write $\mu_n \to \mu$ in this case.

We define the weak* topology on $\mathcal{P}(X)$ as the smallest topology making each of the maps $\mu \to \mu(f) = \int f \, d\mu$ $(f \in C(X))$ continuous.

Proposition 2.1 *Suppose that X is a compact metric space. Then μ_n weakly converges to μ if and only if μ_n converges to μ in the weak* topology. The space $\mathcal{P}(X)$ is metrisable and compact for the weak* topology: If $\mu_n \in \mathcal{P}(X)$ is a sequence then there exists a weakly convergent subsequence.*

A useful way of proving some equidistribution properties is given by Weyl's criterion:

Proposition 2.2 *Let X be a compact metric space. Let $\phi_n \in C(X)$ be a sequence with the property that their linear combinations are dense in $C(X)$ (endowed with the usual norm $\|f\| = \sup_{x\in X} |f(x)|$). Then $\mu_n \to \mu$ if and only if for all $m \in \mathbb{N}$, $\mu_n(\phi_m) \to \mu(\phi_m)$.*

If E is a finite subset of X we define $\mu_E \in \mathcal{P}(X)$ as

$$\mu_E = \frac{1}{|E|} \sum_{x\in E} \delta_x \tag{1}$$

where δ_x denotes the Dirac measure supported at x. We say that a sequence E_n of finite subsets of X is equidistributed for $\mu \in \mathcal{P}(X)$ (or μ-equidistributed) if $\mu_{E_n} \to \mu$.

When X is not compact it's sometimes possible to adapt Weyl's criterion. For modular curves or more generally Shimura varieties, L^2-techniques (spectral decomposition) are used to prove the equidistribution of Hecke points or CM points.

Example 2.3 *Let $X = \mathbb{C}^*$ and E_n be the set of n-th roots of unity. Then E_n is equidistributed for the normalized measure $\frac{d\alpha}{2\pi}$ supported on the unit circle.*

Exercise 2.4 *Prove the last assertion using proposition 2.2. Let n be a integer, ζ_n a primitive n-roots of unity. Let E'_n be the set of Galois conjugate of ζ_n. Using the irreducibility of the cyclotomic polynomials prove that the sequence E'_p (with p a prime number) is $\frac{d\alpha}{2\pi}$-equidistributed. Prove the same result for E'_n.*

2.2 Equidistribution of Galois orbits of algebraic points

Let X be an algebraic projective variety defined over a number field K. For all field L containing K we denote by $X(L)$ the set of L rational points of X. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ and $\mathfrak{G}_K$ the Galois group of $\overline{\mathbb{Q}}$ over K. For all $x \in X(\overline{\mathbb{Q}})$ we define

$$E_x = \{x^\sigma \mid \sigma \in \mathfrak{G}_K\} \tag{2}$$

the Galois-orbit of x.

If we fix an embedding σ of K in $\mathbb{C}$, we can realize E_x as a subset of $X_\sigma(\mathbb{C})$. We write $\Delta_x = \mu_{E_x}$ the associated measure given by (1). A general (unsolved) problem is the following: let x_n be a sequence of points of $X(\overline{\mathbb{Q}})$, what can be said about the weak limits of the associated sequence $\Delta_n = \Delta_{x_n}$ of $\mathcal{P}(X)$.

We are not expecting a general answer to this question but we are going to give significant examples for which it's possible to say something. In all these examples there will be an underlying group structure on the variety X. To avoid some useless pathologies we make the following definition.

Definition 2.5 *Let X be an algebraic variety. A sequence x_n of points of X is said to be "generic" if for all proper algebraic subvariety $Y \subset X$ the set $\{n \in \mathbb{N} \ | x_n \in Y\}$ is finite.*

(Exercise for topologist, prove that a sequence x_n is generic if an only if x_n converges to the generic point of X in the Zariski topology).

2.2.1 The case of $\mathbb{G}_m$: a theorem of Bilu.

The first results (related to the exercise 2.4) is obtained by Bilu. Let $\mathbb{G}_m$ denote the multiplicative group, so $\mathbb{G}_m$ is an algebraic variety defined over $\mathbb{Q}$ and for all field K containing $\mathbb{Q}$ the set $\mathbb{G}_m(K)$ of K-rational points of $\mathbb{G}_m$ is K^*. There is a canonical height function

$$\widehat{h} : \mathbb{G}_m(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_+ \tag{3}$$

satisfying the following conditions:

1. For all $\alpha \in \mathbb{G}_m(\overline{\mathbb{Q}})$ and all $n \in \mathbb{N}$, $\widehat{h}(\alpha^n) = n\widehat{h}(\alpha)$.
2. (Northcott) For all $n \in \mathbb{N}$ and all $X \in \mathbb{R}_+$ the set

$$\{\alpha \in \mathbb{G}_m(\overline{\mathbb{Q}}) \ \mid [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq n \ \text{ and } \widehat{h}(\alpha) \leq X\}$$

   is finite.

A first consequence of these properties is that $\widehat{h}(\alpha) = 0$ if and only if α is a root of unity. In fact $\widehat{h}(1) = \widehat{h}(1^2) = 2\widehat{h}(1) = 0$. If α is a root of unity then there exists a $n \in \mathbb{N}$ such that $\alpha^n = 1$. Hence $\widehat{h}(\alpha^n) = n\widehat{h}(\alpha) = \widehat{h}(1) = 0$. If $\widehat{h}(\alpha) = 0$ then for all $n \in \mathbb{N}$, $\widehat{h}(\alpha^n) = 0$. Applying Northcott we find that $\{\alpha^n, \ \ n \in \mathbb{N}\}$ is a finite set; therefore α is a root of unity.

Another consequence of the Northcott's theorem is that if $\alpha_n \in \mathbb{G}_m(\overline{\mathbb{Q}})$ is a generic sequence of points such that $\widehat{h}(\alpha_n) \to 0$ then $|E_{\alpha_n}| \to \infty$.

Theorem 2.6 *(Bilu[4]) Let $\alpha_n \in \mathbb{G}_m(\overline{\mathbb{Q}})$ be a generic sequence of points such that $\widehat{h}(\alpha_n) \to 0$ then the sets E_{α_n} are $\frac{d\alpha}{2\pi}$-equidistributed.*

Proof. See [4] for this statement and the generalization to the higher rank torus $\mathbb{G}_m^r$.

2.2.2 The case of elliptic curves and abelian varieties.

Let X be an elliptic curve over a field K, so X is an algebraic curve of genus 1 defined over K with the structure of an abelian group on $X(K)$ (we denote by O the neutral element of $X(K)$). If $K = \mathbb{C}$ then X is isomorphic to $\Gamma\backslash\mathbb{C}$ for a lattice $\Gamma \subset \mathbb{C}$. The Lebesgue measure on $\mathbb{C}$ induces a canonical probability measure $\mu_X \in \mathcal{P}(X)$. From this description we see that the set $X[n] = \{P \in E(\mathbb{C}) \ |[n]P = O]\}$ is isomorphic to $\Gamma\backslash\frac{1}{n}\Gamma \simeq \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$.

Exercise 2.7 *Prove that the sequences of subsets $X[n]$ of X is μ_X-equidistributed. (Use proposition 2.2). Let $X'[n]$ be the subset of $X[n]$ consisting of points of order n (for example if p is prime $X'[p] = X[p] - \{O\}$). Prove that the sequence of subsets $X'[n]$ are μ-equidistributed.*

Let X_K be an elliptic curve defined over a field K of characteristic 0. The group $End(X_K)$ of K-endomorphism of X_K is $\mathbb{Z}$ or an order in an imaginary quadratic field. We say that X_K has complex multiplication if $End(X_K) \neq \mathbb{Z}$.

Let X_K be a K-elliptic curve without complex multiplication. A consequence of Serre's open image theorem is that for all p big enough the Galois orbit of a point Q_p of order p is $X'[p]$. With the notation of the section we have $E_{Q_p} = E'[p]$ and therefore the sets E_{Q_p} are μ_X-equidistributed. Using the full strength of Serre's open image theorem it's possible but not obvious to prove that if Q_n is a sequence of points of $X(\overline{\mathbb{Q}})$ with Q_n of order n then the sets E_{Q_n} are μ_X-equidistributed. If X_K has complex multiplication you may have infinitely many prime numbers p such that there exist a point Q_p of $X(\overline{\mathbb{Q}})$ of order p with E_{Q_p} of cardinality $p-1$ (corresponding to a cyclic isogenies of order p).

Using Arakelov theory it is possible to prove a very general result for the equidistribution of Galois-orbits of points with small heights on abelian varieties (containing the equidistribution of the sets E_{Q_n} even for an elliptic curve with complex multiplication.)

An abelian variety A of dimension g over $\mathbb{C}$ is a complex torus $A \simeq \Gamma\backslash\mathbb{C}^g$ endowed with the structure of a projective algebraic variety. The Lebesgue measure on $\mathbb{C}^n$ induces a canonical probability measure $\mu = \mu_A$ on A. We deduce from this description that $A(\mathbb{C})$ is an abelian group. A point P of A is said to be a torsion point if there exists $n \in \mathbb{N}$ such that $[n]P = O$. The set

$$A[n] = \{P \in A(\mathbb{C}) | [n]P = 0\}$$

is isomorphic to $\mathbb{Z}/n\mathbb{Z}^{2g}$ as an abelian group.

More generally an abelian A_K variety over a field K is a projective algebraic variety endowed with the structure of an abelian group structure. Concretely for all extension L of K $A(L)$ is an abelian group. If K is a number field $A(K)$ is finitely generated (Mordell-Weil Theorem).

As in the case $\mathbb{G}_m$, if A_K is defined over a number field K it's possible to define a canonical height function

$$\widehat{h}: \; A(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_+$$

(the Néron-Tate height) with the properties

1. For all $\alpha \in A(\overline{\mathbb{Q}})$ and all $n \in \mathbb{N}$, $\widehat{h}([n]\alpha) = n^2\widehat{h}(\alpha)$.

2. (Northcott) For all $n \in \mathbb{N}$ and all $X \in \mathbb{R}_+$ the set

$$\{\alpha \in A(\overline{\mathbb{Q}}) \ \mid [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq n \text{ and } \widehat{h}(\alpha) \leq X\}$$

   is finite.

Remark 2.8 The height function depends on the choice of a symmetric ample divisor. By definition of a projective variety we can find an ample line bundle $\mathcal{L}$ on A (sometime we say that $\mathcal{L}$ is a polarization). A line bundle $\mathcal{M}$ on A is said to be symmetric if $[-1]^*\mathcal{M} \simeq \mathcal{M}$. One can show that if $\mathcal{L}$ is ample then $[-1]^*\mathcal{L} \otimes \mathcal{L}$ is ample symmetric.

Exercise 2.9 *Let A_K be an abelian variety defined over a number field K. Prove that a point $P \in A(\overline{\mathbb{Q}})$ is torsion if and only if $\widehat{h}(P) = 0$. If P_n is a generic sequence of points of $A(\overline{\mathbb{Q}})$ such that $\widehat{h}(P) \to 0$ then the cardinality of the sets $E_{P_n} = \{P_n^\sigma \ \mid \sigma \in \mathfrak{G}_K\}$ tends to ∞. (Hint: try to imitate the case of $\mathbb{G}_m$). All these facts are independent of the choice made in defining the height.*

The following results is due to Szpiro, Zhang and the author [40].

Theorem 2.10 *Let A_K an abelian variety defined over a number field K. For all embedding $\sigma : \ K \to \mathbb{C}$ we denote by μ_σ the canonical probability measure on $A_\sigma = A_K \otimes_\sigma \mathbb{C} \simeq \Gamma_\sigma \backslash \mathbb{C}^g$. Let P_n be a generic sequence of points of $A(\overline{\mathbb{Q}})$ such that $\widehat{h}(P) \to 0$. Then for all $\sigma : \ K \to \mathbb{C}$ the sets $\sigma(E_{P_n})$ are μ_σ-equidistributed on A_σ.*

The proof uses Arakelov theory (see [40]).

2.2.3 Equidistribution of CM elliptic curves.

The j-invariant establishes a bijection between $\mathbb{C}$ and the set of isomorphism classes of elliptic curves over $\mathbb{C}$. The endomorphism ring $\mathrm{End}(E)$ of an elliptic curve E over $\mathbb{C}$ is either $\mathbb{Z}$ or an order in an imaginary quadratic extension of $\mathbb{Q}$. An elliptic curve is said to be CM (meaning complex multiplication) if $\mathrm{End}(E) \neq \mathbb{Z}$. A complex number x is said to be CM if the corresponding elliptic curve over $\mathbb{C}$ is CM.

Let us recall a few facts about CM elliptic curves. A CM elliptic curve is defined over $\overline{\mathbb{Q}}$. Let K be an imaginary quadratic extension of $\mathbb{Q}$ and $O_K \subset K$ be the ring of integers of K. Any order in O_K is of the form $O_{K,f} = \mathbb{Z} + fO_K$ for a unique integer $f \geq 1$. For $f \geq 1$ let $\Sigma_{K,f}$ be the set of isomorphism classes of pairs (E, α), with E a CM elliptic curve and $\alpha : O_{K,f} \to \mathrm{End}(E)$ an isomorphism of rings. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts transitively on $\Sigma_{K,f}$.

Let $\mathrm{Pic}(O_{K,f})$ be the Picard (or class) group of $O_{K,f}$ and $h = h_{K,f}$. Then the cardinality of $\Sigma_{K,f}$ is h. Let $H_{K,f}$ be the maximal abelian extension of K which is unramified outside f. Then for all $E \in \Sigma_{K,f}$ we have $K(j(E)) = H_{K,f}$ and class field theory gives an isomorphism $\mathrm{Pic}(O_{K,f}) \simeq \mathrm{Gal}(H_{K,f}/K)$. Let $d_E = d_{K,f}$ be the absolute value of the discriminant of $O_{K,f}$. By the Brauer-Siegel theorem we get for all $\epsilon > 0$

$$d_E^{1/2-\epsilon} \ll_\epsilon h = |\Sigma_{K,f}| \ll_\epsilon d_E^{1/2+\epsilon}. \tag{4}$$

The modular group $\mathrm{SL}(2,\mathbb{Z})$ acts properly discontinuously on the upper half plane $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}.$$

The modular curve $Y = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ is the set of isomorphism classes of elliptic curves over $\mathbb{C}$. The Poincaré measure $\frac{dx\ dy}{y^2}$ on $\mathbb{H}$ is $SL(2,\mathbb{R})$-invariant and the volume of a fundamental domain for this measure is finite. (We therefore say that $\mathrm{SL}(2,\mathbb{Z})$ is a lattice of $\mathbb{H}$). Let $d\mu_0 = \frac{3}{\pi}\frac{dx\ dy}{y^2}$ the induced probability measure on $Y \simeq \mathbb{C}$.

The following result is due do to Duke [18]:

Theorem 2.11 *(Duke) As $d_{K,f} \to \infty$ the $\Sigma_{K,f}$ are μ-equidistributed.*

The case of fundamentals discriminants (i. e $f = 1$) is the main result of [18]. The extension to the general case using Hecke operators is given in ([9] th. 2.4).

2.3 Equidistribution of Hecke points.

Let $\mathbb{H}$ be the upper half plane, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and $Y = \Gamma\backslash\mathbb{H}$. Let $d\mu_0$ be the Poincaré metric and $D_0 = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ the associated Laplacian. Let

$L^2(\Gamma\backslash\mathbb{H}, d\mu_0)$ be the space of μ_0-square integrable Γ-invariant functions on $\mathbb{H}$.

The Hecke correspondences T_n on $X(1)$ are defined by

$$T_n.z = \sum_{ad=n} \sum_{0\leq b<d} \frac{az+b}{d}. \tag{5}$$

If $s \in \mathbb{C}$ we define $\sigma_s(n) = \sum_{d/n} d^s$. The degree of T_n as a correspondence is then $\sigma_1(n)$.

We have an induced action on functions on $X(1)$ given by:

$$\overline{T}_n f(z) = \frac{T_n}{\sigma_1(n)} f(z) = \frac{1}{\sigma_1(n)} \sum_{y\in T_n.z} f(y). \tag{6}$$

The following result is proved in [9] theorem 2.1. The proof is also taken from this paper. We just replaced the upper bound for eigenvalues of T_n given in [6] by the better bound obtained recently by Kim and Sarnak [28]. (We therefore just replaced $\frac{5}{28}$ by $\frac{7}{64}$).

Theorem 2.12 *a) For all f in $L^2(X(1), d\mu_0)$ $\overline{T}_n.f$ converges to $\int_{X(1)} f(\zeta)d\mu_0(\zeta)$ in $L^2(X(1), d\mu_0)$. More precisely For all f in $L^2(X(1), d\mu_0)$ and for all $\epsilon > 0$, there exists a constant C_ϵ, (depending only on ϵ), such that:*

$$\|\overline{T}_n f - \int_{X(1)} f(\zeta)d\mu_0(\zeta)\| \leq C_\epsilon n^{-\frac{1}{2}+\frac{7}{64}+\epsilon}\|f\|. \tag{7}$$

b) Let f be a bounded C^∞ function on $X(1)$ such that $D_0 f$ is bounded. For all $\epsilon > 0$,there exists $C_{\epsilon,z,f}$ such that

$$|\overline{T}_n f(z) - \int_{X(1)} f(\zeta)d\mu_0(\zeta)| \leq C_{\epsilon,z,f} n^{-\frac{1}{2}+\frac{7}{64}+\epsilon}. \tag{8}$$

c) For all bounded continuous function f on $X(1)$ and all $z \in X(1)$,we have

$$\lim_{n\to+\infty} \overline{T}_n f(z) = \int_{X(1)} f(\zeta)d\mu_0(\zeta). \tag{9}$$

The convergence is uniform on compact sets.

2.3.1 Spectral decomposition of $L^2(X(1), d\mu_0)$

We have the "spectral decomposition"

$$L^2(X(1), d\mu_0) = \oplus_{n\geq 0} \mathbb{C}[\varphi_n] \oplus \mathcal{E} \tag{10}$$

with φ_n an orthonormal family of eigenfunctions of D_0 with associated eigenvalues $-\lambda_n$ and $\mathcal{E}$ is the continuous part of the spectrum. We write as usual s_n et r_n the complex numbers such that $\lambda_n = s_n(1-s_n) = 1/4 + r_n^2$ (r_n is a real number). It's possible to choose the φ_n eigenvectors for all the Hecke operators T_n . We suppose that this choice is made from now.

The part relative to the continuous spectrum is given by the following isometry:

$$\begin{aligned} E:\ L^2(\mathbb{R}_+) &\to \mathcal{E} \\ h &\mapsto \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} h(t) E_\infty(z, \frac{1}{2} + it)\ dt \end{aligned} \tag{11}$$

here $L^2(\mathbb{R}_+)$ is the set of functions on $\mathbb{R}_+$ square integrable for the Lebesgue measure; $E_\infty(z,s)$ is the Eisenstein series at the cusp ∞, given by the formula

$$E_\infty(z,s) = \frac{1}{2} \sum_{(m,n)=1} \frac{1}{|mz+n|^{2s}}.$$

Let $\alpha \in L^2(X(1), d\mu_0)$ be spectrally decomposed as

$$\alpha(z) = \sum_{n\geq 0} A_n \varphi_n(z) + \int_0^{+\infty} h(t) E_\infty(z, \frac{1}{2} + it)\ dt. \tag{12}$$

Then

$$A_n = (\alpha, \varphi_n) = \int_X \alpha(z) \overline{\varphi}_n(z)\ d\mu_0(z)$$

$$h(t) = \tfrac{1}{2\pi} \int_X \alpha(z) E_\infty(z, \frac{1}{2} - it)\ d\mu_0(z)$$

(at least if α is C^∞ with compact support). The L^2 norm of α is then

$$\|\alpha\|^2 = \sum_n |A_n|^2 + 2\pi \int_0^{+\infty} |h(t)|^2\ dt. \tag{13}$$

If $\alpha(z)$ is C^∞ with compact support then the spectral decomposition (12) is absolutely convergent and is uniformly convergent on compact sets (see [27] theorems (4.7) and (7.3)).

2.3.2 Proof of theorem 2.12

We start the proof of theorem 2.12 in the particular case $f = \varphi_n$ for some $n \in \mathbb{N}$ and the case where f appears in the continuous spectrum of D_0. The general case is then obtained using the spectral decomposition (12).

For $f = \varphi_0$ the constant function the theorem 2.12 is clear as the degree of T_n as a correspondence is $\sigma_1(n)$.

Lemma 2.13 *For all $k \geq 1$, and all $z \in X(1)$:*

$$\lim_{n\to\infty} \overline{T_n}\varphi_k(z) = 0 = \int_{X(1)} \varphi_k d\mu_0. \tag{14}$$

We know that φ_k is an eigenfunction of T_n. We define $\alpha_k(n)$ to be the associated eigenvalues:

$$T_n.\varphi_k = \alpha_k(n)\varphi_k.$$

The Ramanujan-Petersson conjecture predicts that:

$$|\alpha_k(n)| \leq d(n)n^{1/2},$$

where $d(n) = \sigma_0(n)$ is the number of positive divisors of n. The best known result towards the Ramanujan-Peterssonn conjecture is [28]:

$$|\alpha_k(n)| \leq d(n)n^{1/2+\frac{7}{64}}. \tag{15}$$

This improves the result of [6] where the bound with $\frac{5}{28}$ instead of $\frac{7}{64}$ was obtained.

For all $\epsilon > 0$ and n big enough

$$|\overline{T_n}\varphi_k(z)| = |\frac{\alpha_k(n)\varphi_k(z)}{\sigma_1(n)}| \leq n^{-1/2+\frac{7}{64}+\epsilon}|\varphi_k(z))|. \tag{16}$$

Lemma 2.14 *Let f be a function in $\mathcal{E} \cap D(X(1))$. Then*

$$\lim_{n\to\infty} \overline{T_n}f(z) = 0 = \int_{X(1)} f d\mu_0. \tag{17}$$

Proof. There exists a function $h(t) \in L^2(\mathbb{R}_+)$ such that:

$$f(z) = \int_0^\infty h(t)E_\infty(z, \frac{1}{2} + it)dt$$

This last integral is absolutely convergent. We recall that $E_\infty(z,s)$ is an eigenform of the Hecke operators T_n:

$$T_n E_\infty(z,s) = n^s \sigma_{1-2s}(n) E_\infty(z,s).$$

We therefore obtain

$$T_n f(z) = n^{1/2} \int_0^\infty n^{it} \sigma_{-2it}(n) h(t) E_\infty(z, \frac{1}{2} + it) dt. \qquad (18)$$

Therefore for all $\epsilon > 0$ and all $n \gg 0$ we get:

$$|\overline{T}_n f(z)| \leq n^{-1/2+\epsilon} \int_0^\infty |h(t) E_\infty(z,s)| dt. \qquad (19)$$

We can now give the proof of theorem 2.12 Let f be a function in $L^2(X(1), d\mu_0))$. The spectral decomposition of f is written:

$$f(z) = \sum_{k \geq 0} A_k \varphi_k(z) + \int_0^{+\infty} h(t) E_\infty(z, \frac{1}{2} + it)\ dt. \qquad (20)$$

We define

$$J_n = \|\overline{T}_n f - \int_{X(1)} f(\zeta) d\mu_0(\zeta)\|.$$

Then

$$J_n = \| \sum_{k \geq 1} \frac{A_k \alpha_k(n) \phi_k}{\sigma_1(n)} + \frac{n^{\frac{1}{2}}}{\sigma_1(n)} \int_0^\infty n^{it} \sigma_{-2it}(n) h(t) E_\infty(z,s) dt \|.$$

Using (13),we obtain:

$$J_n^2 = \frac{1}{\sigma_1(n)^2} \sum_{k \geq 1} |A_k|^2 |\alpha_k(n)|^2 + 2\pi \frac{n}{\sigma_1(n)^2} \int_0^\infty |h(t)|^2 |\sigma_{-2it}(n)|^2 dt.$$

The proof of the (a) of (2.12) is obtained using the upper bounds for $\alpha_k(n)$ given in the equation (15).

We define $D(X(1))$ as the space of C^∞ bounded functions on $X(1)$ such that $D_0 f$ is bounded.

Let $f \in D(X(1))$ and $z \in X(1)$. Using (16) and(19) we find for all $\epsilon > 0$ and all $n \gg 0$ the upper bound:

$$|\overline{T}_n f(z) - A_0| \leq n^{-1/2+5/28+\epsilon}(\sum_{k\geq 1} |A_k \varphi_k(z)| + \int_0^\infty |h(t) E_\infty(z,s)| dt).$$

This ends the proof of the part (b) of (2.12) as $A_0 = \int_{X(1)} f(z) d\mu_0(z)$ and as $f \in D(X(1))$ the spectral decomposition is absolutely convergent

We finally suppose that $f \in C_0(X(1))$. Let $z \in X(1)$ and $\epsilon > 0$.We can find $\phi \in D(X(1))$such that

$$\sup_{x\in X(1)} |f(x) - \phi(x)| \leq \epsilon.$$

Using part (b) of the theorem 2.12, we know that for all $n \gg 0$:

$$|\overline{T}_n \phi(z) - \int_{X(1)} \phi d\mu_0| \leq \epsilon.$$

We define l_n as $I_n = |\overline{T}_n f(z) - \int_{X(1)} f d\mu_0|$. We therefore obtain that

$$I_n \leq |\overline{T}_n f(z) - \overline{T}_n \phi(z)| + |\overline{T}_n \phi(z) - \int_{X(1)} \phi d\mu_0| + |\int_{X(1)} (\phi - f) d\mu_0| \leq 3\epsilon.$$

This ends the proof of the part (c) of the theorem 2.12.

2.3.3 Higher rank generalization

The result of the previous sections can be generalized to an arbitrary almost simple simply connected linear group $G_{\mathbb{Q}}$ (as $SL(n)_{\mathbb{Q}}$ or $Sp(n)_{\mathbb{Q}}$). A proof using harmonical analysis as in the previous part is given in [8] and [9]. The method gives a convergence rate which is often optimal. This is the case for $SL(n)_{\mathbb{Q}}$ or $Sp(n)_{\mathbb{Q}}$ if $n \geq 3$. Note that optimal results are obtained without using the (unknown) generalized Ramanujan conjecture for parameters of automorphic representation of $SL(n)_{\mathbb{Q}}$ or $Sp(n)_{\mathbb{Q}}$. The extension to arbitrary reductive groups is in general easy. A proof of a slightly more general result (without a convergence rate) is obtained by Eskin and Oh [21] by ergodic methods.

3 The Manin-Mumford and the André–Oort conjecture.

3.1 Abstract form of the conjectures

The Manin-Mumford conjecture about torsion points of abelian varieties and the André-Oort conjecture about CM points on Shimura varieties (ex.: the moduli space of principally polarized abelian varieties, Hilbert modular varieties or product of modular curves...) can abstractly be stated in a unified way. The purpose of this section is to explain these conjectures and the relation with some theorems or conjectures about equidistribution.

Let X be an algebraic variety over $\mathbb{C}$. Let $\mathcal{S}(\mathcal{X})$ be a set of irreducible subvarieties of X. A subvariety $Z \in \mathcal{S}(\mathcal{X})$ is called special and a special subvariety of dimension 0 is called a special point. We say that $\mathcal{S}(\mathcal{X})$ is an admissible set of special subvarieties if :

1. $X \in \mathcal{S}(\mathcal{X})$.
2. For all $Z \in \mathcal{S}(\mathcal{X})$ the set of special points $x \in Z$ is Zariski dense in Z.
3. An irreducible component of an intersection of special varieties is a special variety.

Remark 3.1 *As a consequence of property 3, if W is a subset of $X(\mathbb{C})$ there exists a smallest special subvariety Z_W among special subvarieties containing W.*

The main examples of admissible sets of special subvarieties are:

(i) An abelian variety $X = A$, $\mathcal{S}(A)$ is the set of torsion subvarieties. A torsion subvariety is the translate by a torsion point of an abelian subvariety. The special points are the torsion points.

(ii) A torus $X = T$, $\mathcal{S}(T)$ is the set of torsion subvarieties. A torsion subvariety is the product of a point of finite order by a subtorus. The special points are the points of finite order.

(iii) A Shimura variety $X = S$, $\mathcal{S}(S)$ is the set of subvarieties of Hodge type. A subvariety of Hodge type is an irreducible component of the translate by a Hecke operator of a sub-Shimura variety. The special points are the CM points. This case will be detailed in section 3.3.

Note as a general rule that everything is known in the case of an abelian variety or in the case of a torus but despite recent progress the case of Shimura variety is mainly conjectural. Other situations as mixed Shimura varieties (see [33]) or semi-abelian varieties (see [7], [15]) can be considered. It's possible that other situations coming from variations of Hodge structures could be considered.

Conjecture 3.2 *(Abstract form) There are 2 equivalent ways of formulating the conjecture:*

(a): An irreducible component of the Zariski closure of a set of special points is a special subvariety.

(b) Let Y be an algebraic subvariety of X. There exists special subvarieties $\{Z_1, \ldots, Z_r\}$ with $Z_i \subset Y$ such that if $Z \subset Y$ is a special subvariety then

$$Z \subset \cup_{i=1}^{r} Z_i.$$

The conjecture in this abstract way is certainly too optimistic. For example you could take for X any projective variety of dimension $g \geq 2$ and for $\mathcal{S}(\mathcal{X})$ the union of X and the set of all points of X. (I don't know such a trivial counterexample if we impose that $\mathcal{S}(\mathcal{X})$ is countable in the definition of an admissible set). Nevertheless it may be useful to understand it in this form to see what is really used in the important examples. Note that if $Y \subset X$ is a curve the conjecture predicts that Y is special if and only if Y contains infinitely many special points. Let's prove that the two forms of the conjecture are indeed equivalent:

Let Y be an algebraic subvariety of X and Σ_Y the set of special points contained in Y. Let $\{Z_1, \ldots, Z_r\}$ be the components of the Zariski closure of Σ_Y. If (a) is true then the Z_i are special and have the properties of (b).

Let Σ be a set of special points and Y a component of the Zariski closure. By (b) there exists a finite set $\{Z_1, \ldots, Z_r\}$ of special subvarieties of Y such that all the special subvarieties of Y are contained in one of the Z_i. As Y is the Zariski closure of Σ, $Y \subset \cup_{i=1}^{r} Z_i$ and there exists $i \in \{1, \ldots, r\}$ such that $Y = Z_i$. Therefore Y is special.

The theory is even more interesting when:

(a) The variety X is defined over a number field K and the special points are defined over $\overline{\mathbb{Q}}$.

(b) A special subvariety Z of X is canonically endowed with a probability measure μ_Z such that the Zariski closure of $\mathrm{Supp}(\mu_Z)$ is Z.

Definition 3.3 *An admissible set $\mathcal{S}(X)$ of special subvarieties of X with properties (a) and (b) is said to be strongly admissible.*

The property (a) implies that the special subvarieties of X are defined over number field. (A subvariety containing a dense set of points defined over $\overline{\mathbb{Q}}$ is defined over $\overline{\mathbb{Q}}$). If P is a special point the canonical probability measure on P is $\mu_P = \delta_P$. As in section 2.2 we fix an embedding of K in $\mathbb{C}$ and $X(\overline{\mathbb{Q}})$ is realized as a subset of $X(\mathbb{C}) = X$. Let $E_P = E_{P,K} = \{P^\sigma, \sigma \in \mathfrak{G}_K\}$ and

$$\Delta_P = \Delta_{P,K} = \frac{1}{|E_P|} \sum_{x \in E_P} \delta_y.$$

Definition 3.4 *Let X be a variety and $\mathcal{S}(X)$ a strongly admissible set of special subvarieties. A sequence P_n of points in $X(\mathbb{C})$ is said to be* <u>*strict*</u> *(relatively to $(X, \mathcal{S}(X))$) if for all special subvariety $Z \neq X$ of X the set $\{n \in \mathbb{N} \mid P_n \in Z\}$ is finite.*

The expected equidistribution conjecture is

Conjecture 3.5 *(abstract form) Let X be a variety and $\mathcal{S}(X)$ an admissible set of special subvarieties. Let K be a number field over which X is defined. Let P_n be a strict sequence of special points of $X(\mathbb{C})$ then the sets $E_{P_n,K}$ are μ_X-equidistributed: the associated sequence of probability measure $\Delta_{n,K} = \Delta_{P_n,K}$ weakly converges to μ_X.*

Proposition 3.6 *Conjecture 3.5 implies conjecture 3.2.*

Let Σ be a set of special points and Y a component of the Zariski closure of Σ. Then $\Sigma_Y = \Sigma \cap Y$ is a Zariski dense subset of special points of Y. Let $Z = Z_Y$ be the smallest special subvariety of X containing Y. The set $\mathcal{S}(Z)$ of special subvarieties of X contained in Z is strongly admissible. The subvariety Z is defined over a number field L.

Lemma 3.7 *There exists a strict sequence of special points of Σ_Y (relatively to $(Z, \mathcal{S}(Z))$).*

The set of special subvarieties is countable as special subvarieties are defined over $\overline{\mathbb{Q}}$. We can therefore write $\mathcal{S}(Z) = \{(Z_n),\ n \in \mathbb{N}\}$. For all $n \in \mathbb{N}$ we define

$$\Sigma_{n,Y} = \{P \in \Sigma_Y | P \notin \cup_{i=1}^n Z_i\ \}.$$

As Σ_Y is Zariski dense in Y, for all $n \in \mathbb{N}$, $I_n \neq \emptyset$. We can therefore choose $P_n \in \Sigma_{n,Y}$. By construction P_n is a strict sequence.

Using conjecture 3.5 we see that the sequence $\Delta_{P_n,L}$ weakly converges to μ_Z. As $\mathrm{Supp}(\Delta_{P_n,L})$ is contained in L for all n (and Y is closed) we find that $\mathrm{Supp}(\mu_Z) \subset Y$. As the Zariski closure of $\mathrm{Supp}(\mu_Z)$ is Z (by property (b)), $Y = Z$. Therefore Y is a special subvariety as predicted by conjecture 3.2.

In fact a even more general result is expected. Let X be an algebraic variety defined over a number field K and $\mathcal{S}(X)$ a strongly admissible set of special subvarieties. For all $Z \in \mathcal{S}(X)$, the set

$$O(Z) = \{Z^\sigma \ \mid \sigma \in \mathfrak{G}_K\}$$

is finite and contained in $\mathcal{S}(X)$. Let Δ_Z be the measure

$$\Delta_Z = \frac{1}{|O(Z)|} \sum \mu_{Z_\sigma}.$$

Let $\mathcal{P}(X)$ be the set of Borel probability measure on X and

$$\mathcal{Q}(X) = \{\Delta_Z \ \mid Z \in \mathcal{S}(X)\}.$$

The most optimistic conjecture about equidistribution is:

Conjecture 3.8 *The subset $\mathcal{Q}(X)$ of $\mathcal{P}(X)$ is compact. If Δ_{Z_n} is a sequence of measure in $\mathcal{Q}(X)$ weakly converging to μ_Z then for all $n \gg 0$, $\mathrm{Supp}(\mu_{Z_n}) \subset \mathrm{Supp}(\mu_Z)$.*

We will discuss results for this conjecture for abelian varieties in sections 3.2 and some related results for some sequences μ_{Z_n} where Z_n is a special subvariety (and therefore geometrically irreducible) in section 4.

3.2 The Manin-Mumford and the Bogomolov conjecture.

In the case of abelian varieties, all the abstract theory of the previous section is proved. If A is an abelian variety, a special point is a torsion point and a special variety is a torsion variety. Let $\mathrm{Tor}(A)$ be the set of torsion points of A. The conjecture 3.2 in this case is due to Manin and Mumford:

Theorem 3.9 *Let A be an abelian variety and Σ and X a subvariety of A. Then*

$$X \cap \mathrm{Tor}(A) = \cup_{i=1}^{r} T_i \cap \mathrm{Tor}(A)$$

for some torsion subvarieties $(T_1, \ldots, T_r)$.

A first proof of the conjecture was given by Raynaud [36], (see [39] for the case of a curve) using p-adic method. Hindry [25] gave a proof using Galois theory and diophantine approximation. Hrushowski [26] gave a proof using ideas from logic (model theory of field). As model theory of field is not so far from the theory of constructible set in algebraic geometry it's not completely a surprise that Pink and Roesler [34] where able to translate in a short and efficient way Hrushowski's proof in the language of algebraic geometry (and some Galois theory). Finally a proof using Arakelov theory and ideas from "equidistribution of points with small height" of the Bogomolov conjecture (to be discussed later in this section) was given by Zhang [49] and the author [41]. As the Bogomolov conjecture contains the Manin-Mumford conjecture, this gives an almost completely analytic proof of the Manin-Mumford conjecture.

Recall that if $(a_n)_{n\in\mathbb{N}}$ is a sequence of algebraic points of an abelian variety A defined over a number field we say that (a_n) is a generic sequence (resp. a strict sequence) If for any proper subvariety $Y \subset A$ (resp. for any proper torsion subvariety $Y \subset A$) the set

$$\{n \in \mathbb{N}, a_n \in A(\overline{\mathbb{Q}})\}$$

is finite.

Remark 3.10 With these definitions we can rephrase the Manin-Mumford conjecture in the following way: "Any strict sequence of torsion points of $A(\overline{\mathbb{Q}})$ is generic". If a_n is a strict sequence of torsion points of $A(\overline{\mathbb{Q}})$, Y a proper subvariety of A such that $T_Y = \{n \in \mathbb{N},\ a_n \in Y(\overline{\mathbb{Q}})\}$ is not finite. The Manin-Mumford conjecture implies that the components of the Zariski closure of the a_n with $n \in T_Y$ are torsion subvarieties containing infinitely many terms of the sequence a_n. This contradicts the hypothesis that a_n is strict. The other direction can be proved in the same lines as the proposition 3.6 and is left as an exercise.

When you combine the Manin-Mumford conjecture and the theorem 2.10 you obtain the following results [40] in the direction of the conjecture 3.8:

Theorem 3.11 *Let A_K an abelian variety defined over a number field K. For all embedding $\sigma : K \to \mathbb{C}$ we denote by μ_σ the canonical probability measure on $A_\sigma = A_K \otimes_\sigma \mathbb{C} \simeq \Gamma_\sigma \backslash \mathbb{C}^g$. Let P_n be a strict sequence of torsion points of $A(\overline{\mathbb{Q}})$. Then for all $\sigma : K \to \mathbb{C}$ the sets $\sigma(E_{P_n})$ are μ_σ-equidistributed on A_σ.*

For abelian varieties, the full conjecture 3.8 is a consequence of the extension of this last result to the equidistribution of Galois orbits of special subvarieties due (independently) to Autissier [2] and Baker-Ih [3].

The Bogomolov conjecture is a generalization of the Manin-Mumford conjecture once we recall that a point P of an abelian variety defined over a number field is a torsion point if and only if the Néron-Tate heights $\widehat{h}(P)$ of P is 0 :

Conjecture 3.12 *(Bogomolov) Let A be an abelian variety defined over a number field. Let Y be a non torsion subvariety of A. There exists $c > 0$ such that the set*

$$\{P \in Y(\overline{\mathbb{Q}}) \mid \widehat{h}(P) < c\}$$

is not Zariski-dense in Y.

The idea behind this conjecture is the following. Lang's conjecture predicts that the set of rational points $Y(K)$ of a variety of general type over a number field K should not be Zariski dense in Y. This has been checked by Faltings [23] for non torsion varieties (the case of curve [22] is the celebrated Mordell conjecture). Such a variety certainly contains infinitely many algebraic points but $Y(\overline{\mathbb{Q}})$ is not to big: It's a discrete set in the Néron-Tate topology.

As in the remark 3.10, the Bogomolov conjecture is equivalent to the statement that "any strict sequences a_n of points of $A(\overline{\mathbb{Q}})$ such that $\widehat{h}(a_n) \to 0$ is a generic sequence". The statement of theorem 3.11 remains true with "torsion points" replaced by points with Néron-Tate height tending to 0.

The proof of this conjecture in the case of a curve in its jacobian is given in [41] and the general case is proved along the same lines in [49]. It's unfortunately beyond the scope of these notes to give a detailed account of the proof of the Bogomolov conjecture. The interested reader can read the account given in Bourbaki's seminar by Abbes [1].

Let's just sketch the principle of the proof in the case of a curve in its jacobian. The starting point is a general theorem about the "equidistribution of generic sequences of points with small heights" [40] for more general heights than the Néron-Tate height on abelian varieties.

Let X a curve of genus $g \geq 2$ defined over a number field K and fix an embedding ϕ of X in its jacobian J. The canonical height $\widehat{h}$ on $J(\overline{\mathbb{Q}})$ induces a canonical height on $X(\overline{\mathbb{Q}})$. Fix an embedding of K in $\mathbb{C}$, then $X_{\mathbb{C}}$ is a Riemann surface. We have a natural hermitian inner product on the space $H^0(X_{\mathbb{C}}, \Omega^1_X)$ of holomorphic differential forms on $X_{\mathbb{C}}$ given by

$$(\alpha, \beta) = \frac{i}{2} \sum_X \alpha \wedge \overline{\beta}.$$

Let $\{\omega_1, \ldots, \omega_g\}$ be an orthonormal basis of $H^0(X_{\mathbb{C}}, \Omega^1_X)$. Then we define a canonical $(1,1)$-form μ on $X_{\mathbb{C}}$ by setting

$$\mu := \frac{i}{2g} \sum_{k=1}^{g} \omega_k \wedge \overline{\omega_k}.$$

The form μ does not depend on a choice of an orthonormal basis. The associated measure μ is called the canonical or the Arakelov measure.

Let $P_n \in X_K(\overline{\mathbb{Q}})$ be a generic sequence such that $\widehat{h}(P_n) \to 0$. The result of [40] implies that the associated sequence of Galois orbits (as defined in section 2.2) converges weakly to μ.

Let $\phi_g : X^g \to J$ be the morphism $(x_1, \ldots, x_g) \mapsto \sum_{i=1}^{g} \phi(x_i)$. Let $\pi_i = X^g \to X$. By a diagonal process, it's possible to construct a generic sequence $y_n = (x_{1,n}, \ldots, x_{g,n})$ of $X^g(\overline{\mathbb{Q}})$ such that for all i, $\widehat{h}(x_{i,n}) \to 0$. Using the result of [40], we find that the associated sequence of Galois orbits converges weakly to the measure

$$\mu_g = \pi_1^* \mu \wedge \ldots \wedge \pi_g^* \mu.$$

As $z_n = \phi_g(y_n)$ is a generic sequence of $J(\overline{\mathbb{Q}})$ such that $\widehat{h}(z_n) \to 0$, using theorem 2.10 we know that the associated sequence of Galois orbits converges weakly to the normalized Haar measure μ_J of J.

Combining the two results (and using easy results about the morphism ϕ_g) we obtain the equality:

$$\phi_g^* \mu_J = g! \mu_g = g! \pi_1^* \mu \wedge \ldots \wedge \pi_g^* \mu.$$

It's easy to see that μ_g is everywhere positive and that $\phi_g^* \mu_J$ is 0 at the points where the morphism ϕ_g is singular (for example at $(P_0, \ldots, P_0)$ for a Weierstrass point of $X_{\mathbb{C}}$). This contradiction finishes the proof.

3.3 The André-Oort conjecture.

The André-Oort conjecture is the analogue for Shimura varieties of the Manin-Mumford conjecture for abelian varieties. It's not possible to give here a complete account of Shimura varieties. The interested reader should see [16], [17] or [29] but two aspects should be kept in mind.

1. Shimura varieties are hermitian locally symmetric spaces.
2. Shimura varieties are moduli-spaces for interesting objects as abelian varieties.

The aim of this part is to describe the special points and special subvarieties in this context and to formulate the André-Oort conjecture. We will focus on examples.

Hermitian locally symmetric space.

Let $G = G_{\mathbb{Q}}$ be a connected reductive group over $\mathbb{Q}$, $G(\mathbb{R})^+$ the connected component of 1 of $G(\mathbb{R})$ and K_∞ a maximal compact subgroup of $G(\mathbb{R})$. Let $Z(G)$ be the center of G. Then G is the almost direct product

$$G \simeq Z(G)G_1G_2\dots G_r$$

for some $\mathbb{Q}$-simple groups G_i. We make the following assumption:

(*): For all $i \in \{1\dots,r\}$, $G_i(\mathbb{R})$ is not compact.

The space $X^+ = G(\mathbb{R})/Z(G)(\mathbb{R})K_\infty$ is called a symmetric space. When X^+ is endowed with an $H(\mathbb{R})^+$-invariant complex structure we say that X^+ is an hermitian symmetric space. A couple $(G_{\mathbb{Q}}, X^+)$ is called a (connected) <u>Shimura datum</u>. Deligne [16], [17] proved that such an X^+ is a connected component of the $G(\mathbb{R})$-conjugacy class X of a morphism of algebraic groups

$$\alpha : \quad \mathbb{S} :\longrightarrow G_{\mathbb{R}}$$

Here $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ is the Deligne torus (so $\mathbb{S}$ is $\mathbb{C}^*$ as an algebraic group.) If $x \in X^+$, we'll write $x(\mathbb{S}) \subset G(\mathbb{R})$ for the image of the associated morphism $x : \mathbb{S} \to G_{\mathbb{R}}$. A Shimura datum is defined in [16] as a couple $(G_{\mathbb{Q}}, X)$.

The simple groups $G_{\mathbb{R}}$ such that $X^+ = G(\mathbb{R})/K_\infty$ is hermitian symmetric are well known inside the classification of linear semi-simple non compact groups over $\mathbb{R}$. For example the symplectic group $Sp(2,g)$, unitary groups $U(p,q)$ or orthogonal groups $So(N,2)$ have an associated symmetric space which is hermitian. The symmetric spaces associated to $\mathrm{SL}(n,\mathbb{R})$ $(n \geq 3)$ or $So(p,q)$ (with $p \neq 2$ and $q \neq 2$) are not hermitian.

A subgroup Γ of $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$ is called an arithmetic lattice if Γ is commensurable to $G_{\mathbb{Z}}(\mathbb{Z})$ for a $\mathbb{Z}$-structure on $G_{\mathbb{Q}}$. This notion is independent of a choice of a $\mathbb{Z}$-structure on $G_{\mathbb{Q}}$. A standard way of producing such a $\mathbb{Z}$-structure is to fix an embedding of $G_{\mathbb{Q}}$ in $\mathrm{GL}(n,\mathbb{Q})$ and to take for $G_{\mathbb{Z}}$ the Zariski closure of $G_{\mathbb{Q}}$ in $\mathrm{GL}(n,\mathbb{Z})$.

Any symmetric space $X = G(\mathbb{R})/K_\infty$ is endowed with a $G(\mathbb{R})$-invariant measure. If Γ is an arithmetic lattice, this measure induces a measure on $\Gamma\backslash X^+$. The volume of $S = \Gamma\backslash X^+$ is finite for this measure (hence the notion of lattice—see Borel [5] for a proof). A space of the form $\Gamma\backslash X^+$ for a lattice Γ is called a locally symmetric space. Any locally symmetric space S is therefore endowed with a canonical probability measure μ_S.

If X^+ is hermitian symmetric and Γ is an arithmetic lattice then $S = \Gamma\backslash X^+$ is endowed with a complex structure. Such a S is an (arithmetic) hermitian locally symmetric space. The main fact is the relation with the world of algebraic geometry :

(**Baily-Borel**) There exists a unique structure of algebraic variety on $S = \Gamma\backslash X^+$ over $\mathbb{C}$ such that for any algebraic variety T, any analytic morphism from T to S is induced from a morphism of algebraic varieties. With this structure S is quasi-projective. If Γ is torsion free then S is smooth.

If moreover $\Gamma \subset G(\mathbb{Q})$ is a congruence lattice (Γ contains the Kernel $\Gamma(N)$ of the map $G_{\mathbb{Z}}(\mathbb{Z}) \to G_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z})$ for some $n \in \mathbb{N}$) we say that $S = \Gamma\backslash X^+$ is a "connected Shimura variety".

Example 1 $G = \mathrm{SL}(2,\mathbb{Q})$, $K_\infty = SO_2(\mathbb{R})$, $X^+ = \mathbb{H}$ is the upper half plane. If Γ is a congruence subgroup of $SL(2,\mathbb{Q})$ $S = \Gamma\backslash\mathbb{H}$ is a modular curve and S is a moduli space for elliptic curves with an additional structure defined by Γ.

Example 2 $G = GS_p(2g,\mathbb{Q})$, $K_\infty = U_g(\mathbb{C})$, $X^+ = \mathbb{H}_g$ is the Siegel-Half plane. If $\Gamma = \Gamma(N)$ then $S = \mathcal{A}_{g,N}$ is the moduli space of principally polarized abelian varieties of dimension g with full N- level structure.

Example 3 Let F be a totally real extension of $\mathbb{Q}$ of degree g. $G_{\mathbb{Q}} = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{SL}(2,F)$. Then $G(\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})^g$, $K_\infty = SO_2(\mathbb{R})^g$, $X^+ = \mathbb{H}^g$. Take $\Gamma = \mathrm{SL}_2(O_F)$, then $S = \mathrm{SL}_2(O_F)\backslash\mathbb{H}^g$ is an Hilbert modular variety parametrizing polarized abelian varieties A of dimension g with an imbedding of O_F in the endomorphism of A.

Example 4 Let F be as before and B a quaternion algebra over F. Then $B(\mathbb{R}) = M_2(\mathbb{R})^d \times \mathbf{H}^{g-d}$ (where $\mathbf{H}$ is the usual quaternions over $\mathbb{R}$). Let $G_{\mathbb{Q}}$ be the group of elements of B^* with reduced norm 1 and Γ the units of norm one in B^*. Then $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})^d \times SO_3(\mathbb{R})^{g-d}$, $K_\infty = SO_2(\mathbb{R})^d \times SO_3(\mathbb{R})^{g-d}$ and $X^+ = \mathbb{H}^d$. Then $S = \Gamma\backslash\mathbb{H}^d$ is a "quaternionic Shimura" variety. Example 3 corresponds to $B = M(2, F)$. When $d = 1$, S is a curve (a Shimura curve). Any curve which is a "connected Shimura" variety is obtained from such a quaternion algebra.

Hecke correspondences. Let $S = \Gamma\backslash G(\mathbb{R})^+/K_\infty = \Gamma\backslash X^+$ a connected Shimura variety. Let $q \in G(\mathbb{Q})$, as Γ is an arithmetic lattice $q^{-1}\Gamma q$ is commensurable to Γ: $\Gamma \cap q^{-1} \cap \Gamma q$ is of finite index in Γ and $q^{-1}\Gamma q$. Let $C(\Gamma)$ be the commensurator of Γ

$$C(\Gamma) = \{g \in G(\mathbb{R}) \mid \quad g\Gamma g^{-1} \text{ commensurable with} \Gamma\}.$$

If $G = G^{ad}$ then $C(\Gamma) = G(\mathbb{Q})$, for a general reductive group over $\mathbb{Q}$ see ([35], prop. 4.6 p. 206).

Let q be an element of $C(\Gamma)$. Let $S_q = \Gamma \cap q^{-1}\Gamma q\backslash X^+$ and α_q the finite map $S_q \rightarrow S$ induced from the inclusion $\Gamma \cap q^{-1} \cap \Gamma q \subset \Gamma$. The translation by q on X^+ (given by $x \mapsto g.x$ induces a second finite morphism $\beta_q : \quad S_q \rightarrow S$.

Let T_q be the image in $S \times S$ of S_q by the map (α_q, β_q). Then T_q is an algebraic correspondence on S. Such a correspondence is called a modular correspondence. For all $x \in S$, we have "$T_q.s = \beta_q(\alpha_q^{-1}(x))$" where we have to count with multiplicities the points in $(\alpha_q^{-1}(x))$. If Γ acts on X^+ without fixed points the maps α_q and β_q are unramified (therefore for all $x \in S$, $(\alpha_q^{-1}(x))$ has exactly $deg(\alpha) = [\Gamma \cap q^{-1} \cap \Gamma q : \Gamma]$ points). If $S = \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}$ and $q = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$ for a prime number l then T_q is the usual Hecke operator T_l discussed in section 2.3.

Special subvarieties.

Let S be a connected Shimura variety, we would like to define a set $\mathcal{S}(S)$ of special subvarieties with the properties $(1, 2, 3)$ of section 3.1. A subvariety Z in $\mathcal{S}(S)$ should be (using our two points of view) a sub-hermitian locally symmetric space and a moduli space for objects of S with some additional structures (as polarization, level, endomorphism...).

If (G_1, X_1^+) is a Shimura datum with $G_1 \subset G$ (as $\mathbb{Q}$-algebraic groups) inducing an inclusion $X_1^+ \subset X$, we say that (G_1, X_1^+) is a sub-Shimura datum. $\Gamma_1 = \Gamma \cap G_1(\mathbb{R})^+$ is an arithmetic lattice of G_1. A special subvariety of S is the image S_1 of $\Gamma_1\backslash X_1^+$ for a sub-Shimura-datum (G_1, X_1^+).

For all $x \in X^+$ the subgroup of $G(\mathbb{R})^+$ fixing x is the product of $Z(G)(\mathbb{R}) \cap G(\mathbb{R})^+$ and a maximal compact subgroup K_x of $G(\mathbb{R})^+$. If there exists a torus

$T_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$ such that

$$x(\mathbb{S}) \subset T(\mathbb{R})^{+} \subset Z(G)(\mathbb{R})^{+} K_x$$

then $(T_{\mathbb{Q}}, \{x\})$ is a sub-Shimura datum. We say that x is a special point of X^{+} and its image in S is a special point of S. The set of special points of S is obtained in this way.

The relation with the theory of complex multiplication is the following. A CM-field is a totally imaginary extension of degree 2 of a totally real number field. A simple abelian variety A of dimension g is CM if the endomorphism $\mathrm{EndA} \otimes \mathbb{Q}$ of A is a CM field of dimension $2g$. An abelian A variety is said to be CM if A is isogenous to a product of simple CM abelian varieties. If $G = GS_p(2g, \mathbb{Q})$ (as in example 2) and $x \in \mathbb{H}_g$. Then $x(\mathbb{S})$ is contained in the real points $T(\mathbb{R})^{+}$ of a $\mathbb{Q}$-torus $T_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$ if and only if the image of x in $S = \mathcal{A}_{g,N}$ corresponds to a CM abelian variety. A similar description in terms of endomorphism of Hodge structure exists for a general Shimura variety. Therefore a special point is often called a CM point.

With these definitions one can check that special points are dense in the Zariski topology in any special variety (they are in fact dense in the analytic topology). The existence of one CM point is given by the study of the space $\mathcal{T}_G$ of maximal tori of $G_{\mathbb{Q}}$. Suppose that G is semi-simple. It can be shown that $\mathcal{T}_G$ is a rational variety and that the locus of compact tori is open in the usual topology. Any $\mathbb{Q}$-rational point of $\mathcal{T}_G$ which is in this open set will define a CM point. Note that if $x \in S$ is special then for all $g \in G(\mathbb{Q})$, $T_g.x$ is a finite union of special points and the union of the $T_g.x$, for $g \in G(\mathbb{Q})$ is dense in the analytic topology of S.

A component of the intersection of special subvarieties is special (this is not clear from the point of view of hermitian locally symmetric spaces, but from the moduli point of view the intersection is interpreted as the locus of points with the additional structures of all the subvarieties we are intersecting).

A component of the image by a Hecke operator of a special variety is a special variety.

Example 1 The special subvarieties of $S = \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}$ are S and the CM points corresponding to the CM elliptic curves studied in section 2.3.

Example 2 If S is a Shimura variety any Hecke correspondence T_q is a special subvariety of the Shimura variety $S \times S$.

Example 3 The j-function induces an isomorphism $S = \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H} \simeq \mathbb{C}$. The special subvarieties of $\mathbb{C} \times \mathbb{C}$ are

(i) $S \simeq \mathbb{C}$.

(ii) Couples of CM points .

(iii) Curves of the form $\{x\} \times \mathbb{C}$ (or $\mathbb{C} \times x$) for some CM point x.

(iiii) Modular correspondences $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ associated to $q_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Q})$: $Y_0(N)$ is the image in $\mathbb{C}\times\mathbb{C}$ of the map $(j(\tau), j(N\tau))$. $Y_0(N)$ is the (coarse) moduli space for triples $(E, E', \alpha : \ E \to E')$ where α is a cyclic isogeny of degree N.

Example 4 For each totally real number field F of degree g the associated Hilbert modular varieties are special subvarieties of Siegel modular varieties. This is clear from the "moduli interpretation" and the translation in terms of Shimura data is for example explained in ([44], chap. 9.1) Any component of a Hecke translate of an Hilbert modular variety is again an Hilbert modular variety associated to an order A of O_F.

The conjecture 3.2 in this case was formulated by André for a curve in a general Shimura variety and by Oort for subvarieties of arbitrary dimension in $\mathcal{A}_g$.

Conjecture 3.13 *(André-Oort) (a): An irreducible component of the Zariski closure of a set of CM points is a special subvariety.*

(b) Let Y be an algebraic subvariety of X. There exists special subvarieties $\{Z_1, \ldots, Z_r\}$ with $Z_i \subset Y$ such that if $Z \subset Y$ is a special subvariety then

$$Z \subset \cup_{i=1}^r Z_i.$$

A deep fact of the theory of Shimura varieties is that any Shimura variety is "canonically" defined over a number field. In particular CM points are defined over $\overline{\mathbb{Q}}$. One proof of this fact is given by Faltings [24] using a rigidity argument. Another proof is an important achievement of the work of several mathematicians. Let's just mention among them Shimura, Deligne, Borovoi, Milne, Shih [30]. The fundamental fact which is not given by Faltings's approach is the knowledge of the field of definition (the reflex field) of the Shimura variety which can be computed in terms of the Shimura datum (G, X). Note that if you are allowed to use the adeles (as in Deligne's approach) then the Shimura variety

$$Sh_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

(where K is a compact open subgroup of $G(\mathbb{A}_f)$), is canonically defined over the reflex field. With our definition, a Shimura variety is a connected component of $Sh_K(G, X)$ and is defined over an abelian extension of the reflex field.

Moreover a special subvarieties Z of a Shimura variety is a hermitian locally symmetric space, therefore Z is endowed with a canonical probability measure μ_Z such that $\mathrm{Supp}(\mu_Z) = Z$.

The set $\mathcal{S}(\mathcal{S})$ of special subvarieties is therefore "strongly admissible" (with the terminology of section 3.1). The general equidistribution conjecture can then be stated:

Conjecture 3.14 *The subset $\mathcal{Q}(S)$ of $\mathcal{P}(S)$ is compact. If Δ_{Z_n} is a sequence of measure in $\mathcal{Q}(S)$ weakly converging to μ_Z then for all $n \gg 0$, $\mathrm{Supp}(\mu_{Z_n}) \subset \mathrm{Supp}(\mu_Z)$.*

The evidence for this conjecture is limited. The only known case is the theorem of Duke explained in section 2.2.3: For $S = \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}$, the Galois orbits of CM points are equidistributed. The case of a Shimura curve associated to a quaternion algebra over $\mathbb{Q}$ is almost known by some work of Zhang [50]. General results for equidistribution of the sequence of probability measure associated to <u>connected</u> special subvarieties are obtained using ergodic theory and will be described in section 4.3.

The ideas from the Arakelov theory (successful in the case of Abelian varieties as described in section 3.2) are not applicable here because for any reasonable theory of height, the height of CM points is unbounded (see [13] for the case of CM elliptic curves).

3.3.1 Equidistribution of "Toric orbits" of CM points

No general strategy for general Shimura varieties is known for the conjecture 3.14. The ideas behind Duke's proof of the case of CM elliptic curves can be extended to the case of Hilbert modular varieties ([11], [12]) and [45] or more generally for Shimura varieties associated to quaternion algebra over a totally real number field [50]. This extension unfortunately doesn't lead to the equidistribution of Galois orbits of CM points (and therefore to the André-Oort conjecture) but leads to the equidistribution of a bigger set: the <u>Toric orbits of CM points.</u> Some attempts to obtain in some cases equidistribution of Galois orbits of CM points are given in [45] and [50]. See section 4.2.2.

4 Equidistribution of special subvarieties.

4.1 The case of abelian varieties.

Let $A = \Gamma\backslash\mathbb{C}^n$ be a complex abelian variety. An abelian subvariety B of A is canonically endowed with a probability measure μ_B such that $\mathrm{Supp}(\mu_B) = B$.

Let $\mathcal{P}(A)$ be the set of Borel probability measures on A and

$$\mathcal{Q}(A) = \{\mu_B, \quad \text{with B an abelian subvariety of A}\}.$$

Proposition 4.1 *The set $\mathcal{Q}(A)$ is compact and if μ_{B_n} is a sequence of $\mathcal{Q}(A)$ weakly converging to μ_B then for all $n \gg 0$, B_n is an abelian subvariety of B.*

A sequence B_n of abelian subvarieties of A is said to be strict (relatively to A) if for all proper abelian subvariety B of A the set $\{n \in \mathbb{N} \ \mid B_n \subset B\}$ is finite. For formal reasons the proposition is equivalent to

Proposition 4.2 *Let B_n be a strict sequence of abelian subvarieties of A then μ_{B_n} weakly converges to μ_A.*

Let's recall why the two propositions are equivalent: Let B_n be a sequence of abelian subvarieties. Let $\mathcal{E}$ be the set of abelian subvarieties of A containing infinitely many B_n's. Let A' be a minimal element of $\mathcal{E}$. Replacing B_n by a subsequence we can suppose that B_n is a strict sequence of A' and the proposition 4.2 implies that μ_{B_n} is weakly converging to $\mu_{A'}$. The implication "proposition 4.1 implies proposition 4.2" is simpler (and left as an exercise).

The proof will use only classical Fourier theory. The first step consists in "forgetting the complex structures":

4.1.1 The flat case

In this part we write $G = \mathbb{Q}^n$, $X = \mathbb{Z}^n\backslash\mathbb{R}^n$ and $\pi : \mathbb{R}^n \to$ the canonical morphism. So X is just a C^∞-variety but we define a set $\mathcal{S}(X)$ of special subvarieties by

$$\mathcal{S}(X) = \{Z = \pi(H \otimes_{\mathbb{Q}} \mathbb{R}) \ \text{ with } H \text{ a } \mathbb{Q}\text{-vector subspace of } G\}.$$

Every $Z = \pi(H \otimes_{\mathbb{Q}} \mathbb{R})$ is canonically endowed with a probability measure coming from the Lebesgue measure on $H \otimes_{\mathbb{Q}} \mathbb{R}$.

In this situation we can formulate the analogue of propositions 4.1 and 4.2. As in the previous part the 2 statements are equivalent

The purpose of this part is to prove the analogue of proposition 4.2 :

Proposition 4.3 *Let Z_n be a strict sequence (relatively to the set of special subvarieties) of special subvarieties of X then μ_{Z_n} weakly converges to μ_X.*

For $x \in \mathbb{R}^n$ we write $\overline{x}$ for the class of x in $\mathbb{Z}^n \backslash \mathbb{R}^n$. The set X^* of complex character of X is in bijection with $\mathbb{Z}^n$: For all $\underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ the associated character $\chi_{\underline{k}}$ of X is defined by

$$\chi_{\underline{k}}(\overline{x}_1, \dots, \overline{x}_n) = \exp(2i\pi \sum_{j=1}^{n} k_j x_j).$$

Any character of X is obtained in this way.

If $\chi = \chi_{k_1,\dots,k_n}$ for some $(k_1, \dots, k_n) \in \mathbb{Z}^n - \{(0, \dots, 0)\}$, we write

$$H_\chi = H_{k_1,\dots,k_n}$$

the $\mathbb{Q}$-hyperplane of G defined by

$$\sum_{j=1}^{n} k_j x_j = 0.$$

Then $H_\chi = H_{\chi'}$ with $\chi = \chi_{k_1,\dots,k_n}$ and $\chi' = \chi_{k'_1,\dots,k'_n}$ if and only if there exists $\alpha \in \mathbb{Q}$ such that $k'_i = \alpha k_i$ for all i.

Let

$$S_\chi = S_{k_1,\dots,k_n} = \pi(H_{k_1,\dots,k_n} \otimes \mathbb{R})$$

be the associated maximal special subvariety of X. All the maximal special subvarieties are obtained in this way. We define also

$$\tilde{S_\chi} = H_{k_1,\dots,k_n} \otimes \mathbb{R}.$$

Lemma 4.4 *Let S be a special subvariety of X and $\chi = \chi_{k_1,\dots,k_n}$ a non trivial character of X. The restriction χ_S of χ on S is a character and $\chi_S = 1$ if and only if $S \subset S_\chi$.*

The character χ is trivial on S_χ, therefore if $S \subset S_\chi$ then $\chi_S = 1$. If $\chi_S = 1$, $S = \pi(\tilde{S})$ for a $\mathbb{R}$-subvector space of $G(\mathbb{R})$ and $x = (x_1, \dots, x_n) \in \tilde{S}$ then for all $t \in \mathbb{R}$, $t \sum_{i=1}^n k_i x_i \in \mathbb{Z}$. Therefore $\sum_{i=1}^n k_i x_i = 0$ and $\pi(x) \in S_\chi$ and $S \subset S_\chi$.

We can now give a proof of proposition 4.3. Let T_n be a strict sequence of special subvarieties of X and μ_n the associated sequence of probability measure. Using Weyl's criterion, we must show that for all non trivial character of X

$$\lim_{n \to \infty} \int_{T_n} \chi \ d\mu_n = \lim_{n \to \infty} \int_{T_n} \chi_{T_n} \ d\mu_n = 0.$$

Fix a character χ. As T_n is a strict sequence, for all n big enough T_n is not contained in S_χ. By lemma 4.4 χ_{T_n} is a non-trivial character of T_n and

$$\int_{T_n} \chi_{T_n} \ d\mu_n = 0.$$

Exercise 4.5 *Prove the following analogue of 4.1: let S_n be a sequence of special subvarieties of $X = \mathbb{Z}^n \backslash \mathbb{R}^n$. Then there exists a special subvariety S and a subsequence S_{n_k} such that μ_{n_k} converges weakly to μ_S. Moreover for all $k \gg 0$, $S_{n_k} \subset S$.*

Remark 4.6 We can replace $\mathbb{Z}^n$ by an arbitrary lattice Γ of $\mathbb{R}^n$ in the previous statements. There exists a linear automorphism u_Γ of $\mathbb{R}^n$ such that $u_\Gamma(\mathbb{Z}^n) = \Gamma$. Such an automorphism induces an isomorphism of C^∞ varieties;

$$\overline{u_\Gamma} : \ \mathbb{Z}^n \backslash \mathbb{R}^n \longrightarrow \Gamma \backslash \mathbb{R}^n.$$

The special subvarieties of $\Gamma \backslash \mathbb{R}^n$ are just the images by $\overline{u_\Gamma}$ of the special subvarieties of $\mathbb{Z}^n \backslash \mathbb{R}^n$.

We can now give a proof of 4.2. Let $A = \Gamma \backslash \mathbb{C}^n$ an abelian variety. Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and applying the previous remark we have a notion of special subvarieties of A. (Warning: this notion is not the usual one for abelian varieties). With this notion, abelian subvarieties of A are special subvarieties but there exists much more special subvarieties corresponding to non-complex subtori. (For example if A is a simple abelian variety, there exists no sub-abelian varieties).

Let A_n be a sequence of abelian subvarieties of A and μ_n the associated sequence of probability measures. Then using exercise 4.5, there exists a special subvariety S of A and a subsequence μ_{n_k} weakly converging to μ_S. For all k big enough $A_{n_k} \subset S$. We therefore just need to prove that S is an abelian subvariety. The group generated by the A_{n_k} for $k \gg 0$ is generated by finitely many abelian subvarieties and is therefore an abelian subvariety B. So $B \subset S$, but the supports of the μ_{n_k} for $k \gg 0$ are contained in B which is a closed subvariety. As $\mu_{n_k} \to \mu_S$, we find that $S = \mathrm{supp}(\mu_S) \subset B$ and therefore that $S = B$ is an abelian subvariety of A.

4.2 Equidistribution of algebraic measures

Let $G_{\mathbb{Q}}$ be a connected algebraic group over $\mathbb{Q}$ and $X^*(G_{\mathbb{Q}})$ be the set of rational characters of $(G_{\mathbb{Q}})$. We say that $G_{\mathbb{Q}}$ is of type $\mathcal{F}$ if $X^*(G_{\mathbb{Q}}) = \{1\}$. Fix a $\mathbb{Z}$-structure $G_{\mathbb{Z}}$ on $G_{\mathbb{Q}}$ (for example the Zariski closure of $G_{\mathbb{Q}}$ in $GL(n, \mathbb{Z})$ for a faithful representation of G in $GL(n, \mathbb{Q})$). A subgroup $\Gamma \subset G(\mathbb{Q})^+$ is said to be an "arithmetic lattice" if Γ is commensurable with $G_{\mathbb{Z}}(\mathbb{Z})$ (this doesn't depends of the choice of the $\mathbb{Z}$-structure on $G_{\mathbb{Q}}$).

Let G denote the real Lie group $G = G(\mathbb{R})^+$ and let μ_G be the G-invariant measure on $X^+ = \Gamma\backslash G$. Then the μ_G-volume of X^+ is finite, hence the name lattice, (see [35] thm. 4.13, p. 213).

If $H_{\mathbb{Q}} \subset G_{\mathbb{Q}}$ is a connected $\mathbb{Q}$-algebraic subgroup of type $\mathcal{F}$, then $\Gamma_H = \Gamma \cap H(\mathbb{R})^+$ is an arithmetic lattice of $H = H_{\mathbb{Q}}(\mathbb{R})^+$ and

$$X_H^+ = \Gamma\backslash\Gamma H(\mathbb{R})^+ \simeq \Gamma_H\backslash H(\mathbb{R})^+$$

is a closed subset of X^+ endowed with a canonical H-invariant probability measure μ_H. Such a subset is said to be special in this section. A probability measure on X^+ is said to be "algebraic" (or homogeneous) if it is obtained in this way.

Let $\mathcal{P}(X)^+$ be the set of probability measures on X^+ endowed with the weak star topology. Let $\mathcal{Q}(X^+)$ be the subset of $\mathcal{P}(X^+)$ consisting of the algebraic measures. There is a natural definition of strict sequence of $\mathbb{Q}$-subgroups of $G_{\mathbb{Q}}$: such a sequence $H_{n,\mathbb{Q}}$ is said to be strict if for all proper $\mathbb{Q}$-subgroup $H_{\mathbb{Q}}$ the set

$$\{n \in \mathbb{N},\ H_{n,\mathbb{Q}} \subset H_{\mathbb{Q}}\}$$

is finite.

We'll give some general examples of such strict sequence $H_{n,\mathbb{Q}}$ verifying the following equidistribution property:

($\mathcal{E}$) The associated sequence of probability measures $\mu_n = \mu_{H_n}$ weakly converges to μ_G.

The following example shows that the property ($\mathcal{E}$) is not always verified:

Example 4.7 Let $G_{\mathbb{Q}}$ be the group $SU(2)$. Therefore $G = G(\mathbb{R})^+$ is compact and any (arithmetic) lattice Γ is finite. Therefore we may assume that $\Gamma = \{1\}$ and $X^+ = G$. Fix a $\mathbb{Q}$-torus $T_{0,\mathbb{Q}}$ of $G_{\mathbb{Q}}$, and a sequence $g_n \in G(\mathbb{Q})$ converging to $g \in G(\mathbb{R})$. Note that the canonical measure μ_{T_0} is just the normalized Haar measure on $T_0(\mathbb{R})^+$ in this situation. Suppose that the sequence $T_{n,\mathbb{Q}} = g_n T_{0,\mathbb{Q}} g_n^{-1}$ is strict (as an exercise prove that this is possible). Then μ_{T_n} is weakly convergent to the normalized Haar measure on $gT^0(\mathbb{R})^+g^{-1} \neq \mu_G$.

We proposed in [11] (with Laurent Clozel) the following conjecture for which we don't know any counter-example. The formulation uses the adeles. One of the assignments of the organizers was to "avoid the adeles like the plague". So if you are afraid of contamination you should avoid this part.

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ the ring of finite adeles. Let $G_{\mathbb{Q}}$ be an algebraic group of type $\mathcal{F}$. A congruence subgroup of $G(\mathbb{Q})$ is an arithmetic lattice Γ of $G(\mathbb{R})^+$ of the form

$$\Gamma = G(\mathbb{Q})^+ \cap K$$

for an open compact subgroup K of $G(\mathbb{A}_f)$. Let Γ be a congruence subgroup of $G(\mathbb{Q})$ and $X = \Gamma \backslash G(\mathbb{R})^+$. Then X is a component of

$$S(G, K) = G(\mathbb{Q})^+ \backslash G(\mathbb{R})^+ \times G(\mathbb{A}_f)/K$$

and the components of $S(G, K)$ are indexed by the finite set $G(\mathbb{Q})^+ \backslash G(\mathbb{A}_f)/K$ (which should be thought of as a class-group).

If $H_{\mathbb{Q}}$ is a $\mathbb{Q}$-subgroup of type $\mathcal{F}$, then $K_H = H(\mathbb{A}_f) \cap H$ is an open compact subgroup of $H(\mathbb{A}_f)$. Every irreducible component of

$$S(H, K) = S(H, K_H) = H(\mathbb{Q})^+ \backslash H(\mathbb{R})^+ \times H(\mathbb{A}_f)/K_H$$

is endowed with a canonical probability measure. Let $\Theta_{X,H}$ be the set of components of $S(H, K_H)$ which are contained in X and let $h_a = |\Theta_{X,H}|$. The adelic probability measure $\mu_{a,H}$ associated to H is by definition:

$$\mu_{a,H} = \frac{1}{h_a} \sum_{\gamma \in \Theta_{X,H}} \mu_\gamma$$

where μ_γ is the canonical probability measure on the component Z_γ of $S(H, K_H)$ indexed by $\gamma \in \Theta_{X,H}$. The adelic equidistribution conjecture is then:

Conjecture 4.8 *Let H_n be a strict sequence of $\mathbb{Q}$-subgroups of $G_\mathbb{Q}$ of type $\mathcal{F}$. Then the associated sequence of measures μ_{a,H_n} weakly converges to μ_G.*

In the example 4.7 it can be shown that the cardinality of the class group $T_n(\mathbb{Q})^+\backslash T_n(\mathbb{A}_f)/K \cap T_n(\mathbb{A}_f)$ tends to ∞ as n tends to ∞.

4.2.1 Ergodic theory and property $\mathcal{E}$

In this section we explain a general situation where the property $\mathcal{E}$ is verified. We start by the following definition:

Definition 4.9 A connected linear algebraic group over $\mathbb{Q}$ is said to be of type $\mathcal{H}$ if its solvable radical is unipotent and if $H_s = H/R_u(H)$ is an almost direct product of $\mathbb{Q}$-simple groups H_i such that $H_i(\mathbb{R})$ is not compact.

Theorem 4.10 *Let $G_\mathbb{Q}$ be a semi-simple group of type $\mathcal{H}$, and $H_n \subset G_\mathbb{Q}$ be a strict sequence of subgroups of type $\mathcal{H}$. Then the property ($\mathcal{E}$) is verified for the associated sequence of measure $\mu_n = \mu_{H_n}$.*

Let's give some ideas of the proof of such a result. Let $G_\mathbb{Q}$ be as in the statement of the theorem and $\Gamma \in G(\mathbb{Q})^+$ be an arithmetic lattice and $X = \Gamma\backslash G(\mathbb{R})^+$.

Definition 4.11 Let $F \subset G(\mathbb{R})^+$ be a connected closed Lie subgroup. We say that F is of type $\mathcal{K}$ if

(i) $F \cap \Gamma$ is a lattice in F.

Therefore $F \cap \Gamma\backslash F$ is a closed subset of $\Gamma\backslash G(\mathbb{R})^+$. Let μ_F be the associated F-invariant probability measure.

(ii) The subgroup $L(F)$ generated by the unipotent one parameter subgroup of F acts ergodicaly on $F \cap \Gamma\backslash F$ with respect to μ_F. By definition this means that any $L(F)$-invariant measurable subset of $F \cap \Gamma\backslash F$ is of μ_F-measure 0 or 1.

The relation between the class $\mathcal{K}$ and the class $\mathcal{H}$ is given by the following lemma (see [11] for a proof).

Lemma 4.12 *(a) If $H_{\mathbb{Q}}$ is an algebraic $\mathbb{Q}$-sub-group of type $\mathcal{H}$ then $H(\mathbb{R})^+$ is a Lie sub-group of type $\mathcal{K}$*

(b) If F is a closed Lie subgroup of type $\mathcal{K}$, then there exists an algebraic $\mathbb{Q}$-subgroup $F_{\mathbb{Q}}$ of type $\mathcal{H}$ such that $F = F(\mathbb{R})^+$.

The algebraic group $F_{\mathbb{Q}}$ associated to F in this last statement is the "Mumford-Tate group of F": $F_{\mathbb{Q}}$ is the smallest $\mathbb{Q}$-subgroup $H_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$ such that $F \subset H(\mathbb{R})^+$.

A deep result of Ratner [37], [38] (conjectured by Raghunathan) implies that if L is a closed Lie subgroup of $G(\mathbb{R})^+$ generated by one parameter unipotent subgroups then the Mumford-tate group $F_{\mathbb{Q}}$ of L is of type $\mathcal{H}$ and the closure $\overline{\Gamma\backslash\Gamma L}$ in the analytic topology of $\Gamma\backslash\Gamma L$ is $\Gamma\backslash\Gamma F(\mathbb{R})^+ \simeq F(\mathbb{R})^+ \cap \Gamma\backslash F(\mathbb{R})^+$.

Let $\mathcal{P}(X)$ be the set of Borel probability measure on X and $\mathcal{Q}(X)$ be the subset of $\mathcal{P}(X)$ defined as

$$\mathcal{Q}(X) = \{\mu_F, \quad F \in \mathcal{K}\}.$$

As a consequence of the previously discussed work of Ratner, Mozes-Shah [32] proved the following (deep) analogue of 4.1:

Theorem 4.13 *(Mozes-Shah)*

The set $\mathcal{Q}(X)$ is compact in the weak star topology. If μ_n is a sequence of $\mathcal{Q}(X)$ weakly converging to μ, then $\mu \in \mathcal{Q}(X)$ and for all n big enough $Supp(\mu_n) \subset Supp(\mu)$.

The proof of theorem 4.10 is now straightforward. Let $H_{n,\mathbb{Q}}$ be a sequence of algebraic subgroups of $G_{\mathbb{Q}}$ of type $\mathcal{H}$ and $\mu_n \in \mathcal{Q}(X)$ be the associated sequence. If μ_α is a subsequence converging to μ, then $\mu = \mu_H$ for a closed connected Lie sub-group of type $\mathcal{K}$. Then $H = H_{\mathbb{Q}}(\mathbb{R})^+$ for an algebraic $\mathbb{Q}$-subgroup of type $\mathcal{H}$. For all $\alpha \gg 0$, $Supp(\mu_\alpha) \subset \mathrm{supp}(\mu)$, therefore $Lie(H_\alpha(\mathbb{R})) \subset Lie(H(\mathbb{R}))$. Hence $H_\alpha(\mathbb{R})^+ \subset H(\mathbb{R})^+$ and by the definition of the Mumford-Tate group $H_{\alpha,\mathbb{Q}} \subset H_{\mathbb{Q}}$. As the sequence $H_{n,\mathbb{Q}}$ is strict $H_{\mathbb{Q}} = G_{\mathbb{Q}}$ and $\mu = \mu_G$.

4.2.2 Adelic equidistribution for $PGL(2, F)$.

In view of the last section the property $\mathcal{E}$ may fail for sequences $H_{n,\mathbb{Q}}$ of reductive non semi-simple algebraic subgroups of $G_{\mathbb{Q}}$. The following statement ([11] theorem 7.1) is an important case where the property $\mathcal{E}$ may fail, but the adelic equidistribution conjecture 4.8 holds.

Theorem 4.14 *Let F be a number field and $G_{\mathbb{Q}} = Res_{F/\mathbb{Q}}PGL(2,F)$. Let O_F be the ring of integers of F and d_n be a sequence of square free elements of O_F. Then*

$$T'_{n,\mathbb{Q}} = \Big\{ \begin{pmatrix} a & b \\ d_n b & a \end{pmatrix}, \quad a^2 - d_n b^2 \neq 0 \Big\}$$

is a torus of $Res_{F/\mathbb{Q}}GL(2,F)$. Let $T_{n,\mathbb{Q}}$ be the image of $T'_{n,\mathbb{Q}}$ in $G_{\mathbb{Q}}$. Let Γ be a congruence subgroup of $G(\mathbb{Q})^+$ and $X = \Gamma\backslash G(\mathbb{R})^+$. If the norm $N_{F/\mathbb{Q}}(d_n)$ of d_n tends to ∞, then the associated adelic measure $\mu_{a,n} = \mu_{a,T_n}$ weakly converges to μ_G.

We only give a description of the proof which is an extension of Duke's method for the equidistribution of CM points on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ discussed in section 2.2.3.

Let f be a parabolic form on X, π the associated automorphic representation. For $d \in O_F$ with d square free, we denote by Π_d the base change of π to $E_d = F[\sqrt{d}]$. Using a formula of Waldspurger ([46], proposition 7, p. 222) we obtain a relation between $\mu_{a,d}(f)$ and $L(\Pi_d, \frac{1}{2})$ and we show that for all $\epsilon > 0$

$$|\mu_{a,d}(f)| \ll |N_{F/\mathbb{Q}}(d)|^{-\frac{1}{4}+\frac{\theta}{2}+\epsilon}, \tag{21}$$

where θ denotes the "Selberg constant" ($0 \leq \theta < \frac{1}{2}$) measuring the lack of validity of the Selberg conjecture (predicting $\theta = 0$). The Lindelöf hypothesis combined with the Selberg conjecture would give:

$$|\mu_{a,d}(f)| \ll |N_{F/\mathbb{Q}}(d)|^{-\frac{1}{2}+\epsilon}.$$

The same kind of results is obtained for Eisenstein series $E_\chi(g,s)$ associated with a character χ of $F^*\backslash\mathbb{A}_f^*$ using a result of Wielonski [47]: $\mu_{a,d}(E_\chi(g, \frac{1}{2}+ i\sigma))$ is related to the special value of a L-function "à la Tate" $L(\chi N_{E_d/F}, \frac{1}{2})$.

There exists $A > 0$ such that for all $\epsilon > 0$ and all $\sigma \in \mathbb{R}$:

$$|\mu_{a,d}(E_\chi(g, \frac{1}{2} + i\sigma))| \ll |N_{F/\mathbb{Q}}(d)|^{-\frac{1}{4}+\epsilon}|\sigma|^A. \tag{22}$$

The Lindelöf hypothesis for $L(\chi \circ N_{E/F}, \frac{1}{2})$ would give

$$|\mu_{a,d}(E_\chi(g, \frac{1}{2} + i\sigma))| \ll |N_{F/\mathbb{Q}}(d)|^{-\frac{1}{2}+\epsilon}|\sigma|^A.$$

Note that we don't need a subconvexity result in the proof of the theorem. The method of the proof leads to a conditional statement for the analogue statement on the symmetric space. In this case we lose a power of $N_{F/\mathbb{Q}}(d)$ the bounds for $|\mu_{a,d}(f)|$ and $|\mu_{a,d}(E_\chi(g, \frac{1}{2}+i\sigma))|$ and we need a subconvexity bound as in Duke's theorem.

For example if F is totally real, and T_n is a sequence of tori such that $T(\mathbb{R})$ is compact this is the problem of equidistribution of toric orbits of CM points on a Hilbert modular variety discussed briefly in section 3.3.1. Venkatesh [45] has a method which leads to unconditional results in this case.

Note that from the harmonical analysis point of view the situation is much harder than in the case of Duke (say $F = \mathbb{Q}$). Using equations (21) and (22) you get a L^2-convergence and you want to deduce from this a pointwise convergence (see section 2.3 for a detailed similar example). In Duke's case the continuous part of $L^2(SL(2,\mathbb{Z})\backslash\mathbb{H}, d\mu_0)$ is obtained using one Eisenstein series. For a general number field you need to consider an infinite set of Eisenstein series (essentially parametrised by the units of O_F). You therefore need to understand the dependence in χ of the bounds given in the equation (22).

4.3 Equidistribution of special subvarieties of Shimura varieties

The references for this part are [10] and [43].

Let S be a connected Shimura variety as defined in section 3.3. Let Z_n be a sequence of special subvarieties of S. You can't expect in general that the associated sequence of probability measure $\mu_n = \mu_{Z_n}$ weakly converges. For example if x_n is a sequence of CM points then μ_n is just the Dirac measure supported at x_n, such a sequence can converge to δ_x for a non CM point or x_n may tend to ∞. Even for positive dimentional special subvarieties the same problem may occur. Start with a special subvariety $Z \times Z'$ of S for two special varieties Z and Z'. If x_n is a CM point of Z' and $Z_n = Z \times \{x_n\}$ there is no hope of proving the week convergence of μ_n. A special subvariety Z of S is said to be NF (non factor) if Z is not of the form $Z_1 \times \{x\}$ with Z_1 special and x a CM point. The following theorem is obtained in [43] and is a generalization of the main result of [10] obtained with L. Clozel.

Theorem 4.15 *Let $\mathcal{P}(S)$ be the set of Borel probability measure on S. Let $\mathcal{Q}(S) = \{\mu_Z |\ Z \ \ NF\}$. Then $\mathcal{Q}(S)$ is compact and if μ_n is a sequence of $\mathcal{Q}(S)$ converging to μ, then $\mu = \mu_Z \in \mathcal{Q}(S)$ and for all $n \gg 0$, $Z_n \subset Z$.*

As usual, we get the following result in the direction of the André-Oort conjecture.

Theorem 4.16 *Let $Y \subset S$ be a subvariety of a Shimura variety S. Then there exists a finite set $\{Z_1, \dots, Z_r\}$ of NF special subvarieties of Y such that if Z is a special NF subvariety of Y then*

$$Z \subset \cup_{i=1}^{r} Z_i.$$

Let's recall how theorem 4.15 implies 4.16. Let Z_n be a sequence of distinct NF special subvarieties of Y which are maximal among NF special subvarieties of Y. By theorem 4.15 we can suppose that the associated sequence of probability measure μ_n converges weakly to μ_Z for a NF special subvariety of S. But as Y is closed $\mathrm{supp}(\mu_Z) = Z \subset Y$. For n big enough we know that $Z_n \subset Z \subset Y$.

Let's give a sketch of the proof of the theorem 4.15 (the reader should compare with the proof of the proposition 4.1). There exits a semi-simple group $G_{\mathbb{Q}}$ such that the associated symmetric space $\mathcal{D} = G(\mathbb{R})^+/K_\infty$ is hermitian and an arithmetic lattice $\Gamma \in G(\mathbb{Q})^+$ such that $S = \Gamma\backslash\mathcal{D}$.

Let $\Omega = \Gamma\backslash G(\mathbb{R})^+$. We defined in section 4.1.1, a class $\mathcal{H}$ of algebraic $\mathbb{Q}$-subgroups of $G_{\mathbb{Q}}$ and a compact subset $\mathcal{Q}(\Omega)$ of the set $\mathcal{P}(\Omega)$ of Borel probability measure on Ω. A special subvariety Z of S is associated to a $\mathbb{Q}$-subgroup $H_{\mathbb{Q}}$ such that $H_{\mathbb{Q}}^{der}$ is of type $\mathcal{H}$.

Let Z_n be a sequence of special subvarieties and $H_{n,\mathbb{Q}}$ be the associated sequence. Suppose for simplicity that $H_{n\mathbb{Q}}$ is semi-simple (this is the case considered in [10]), then $H_{n\mathbb{Q}}$ is of type $\mathcal{H}$ Using the results of Mozes and Shah (theorem 4.13), we may assume that the associated sequence μ_n of $\mathcal{Q}(\Omega)$ weakly converges to a measure $\mu_H \in \mathcal{Q}(\Omega)$ for an algebraic $\mathbb{Q}$-subgroup $H_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$ of type $\mathcal{H}$. For all $n \gg 0$ we know moreover that $H_{n,\mathbb{Q}} \subset H_{\mathbb{Q}}$. We then show that $H_{\mathbb{Q}}$ is related to Shimura varieties: $H_{\mathbb{Q}}$ should be reductive (and in fact semi-simple in view of the definition 4.9 of type $\mathcal{H}$) and the symmetric space associated to $H_{\mathbb{R}}$ should be of hermitian type. (The formalism of Deligne [16] and [17] is used in this part).

You need then to pass from this result on $\Omega = \Gamma\backslash G(\mathbb{R})^+$ to a result on the Shimura variety $S = \Gamma\backslash G(\mathbb{R})^+/K_\infty$. The main difficulty is the following: for each point $x \in \mathcal{D}$ we have an associated maximal compact subgroup K_x of $G(\mathbb{R})^+$ and a morphism $\pi_x : \Omega \to S$. Let Z be a special subvariety of S with associated canonical probability measure μ_Z. Let $H_{\mathbb{Q}}$ an associated

$\mathbb{Q}$-subgroup and μ_H as previously. To understand the relation between μ_Z and μ_H, you must fix a maximal compact subgroup K_x of $G(\mathbb{R})^+$ such that $H(\mathbb{R})^+ \cap K_x$ is a maximal compact subgroup of $H(\mathbb{R})^+$. Then $\mu_Z = \pi_{x\star}\mu_H$. If you could fix a x such that $K_x \cap H_n(\mathbb{R})^+$ is a maximal compact subset of $H_n(\mathbb{R})^+$ for all $n \in \mathbb{N}$ then the result on Ω would directly imply the theorem 4.16. To overcome this difficulty (which is not serious if Γ is a cocompact lattice of $G(\mathbb{R})^+$– i.e. if $G_{\mathbb{Q}}$ is $\mathbb{Q}$-anisotropic), we use some results of Dani and Margulis on the behavior of unipotent flows [14].

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